INTEGRAL REPRESENTATIONS IN TOPOLOGICAL SPACES

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Abstract Let X, Y be Banach spaces (or either topological vector spaces) and let us consider the function space $C(S, X)$ of all continuous functions $f : S \to X$, from the compact (locally compact) space S into X, equipped with some appropriate topology. Put $C(S, X) = C(S)$ if $X = \mathbb{R}$. In this work we will mainly be concerned with the problem of representing linear bounded operators $T : C(S, X) \to Y$ in an integral form: $f \in C(S, X)$, $Tf = \int_S f \, d\mu$, for some integration process with respect to a measure μ on the Borel $\sigma-$field $\mathcal{B}_S$ of S. The prototype of such representation is the theorem of F. Riesz according to which every continuous functional $T : C(S) \to \mathbb{R}$ has the Lebesgue integral form $Tf = \int_S f \, d\mu$. This paper is intended to present various extensions of this theorem to the Banach spaces setting alluded to above, and to the context of locally convex spaces.

[0](2010)Primary 28C05, Secondary 46G12

[0]*Keywords and Phrases.* Riesz Theorem; Bounded Operators; Integral Representation; Integration Process; Vector Measures; Pettis Integral.

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Introduction

The Integral Representation Theorem of Riesz asserts that if S is a compact Hausdorff space, and $C(S)$ is the space of continuous real-valued functions on S, with the uniform norm, then for each bounded linear functional $T : C(S) \to \mathbb{R}$, there is a unique bounded regular measure μ on the Borel $\sigma-$field $\mathcal{B}_S$ such that:

$$Tf = \int_S f\,d\mu, \text{ for all } f \in C(S)$$

where the integral is a Lebesgue one, and $\|T\| = |\mu|$, the variation of μ.
This statement of the theorem, due to Kakutani $[11]$, is one of the versions we have at hand nowadays.

A lot of work has been done on various extensions of this theorem (see $[1], [4], [5], [16], [18]$ and the references therein.), and it is still the object of many investigations $[2], [13], [20]$. The aim of this work is intended to present a unified survey of some prominent generalizations of Riesz Theorem, frequently used in the literature. The paper has, obviously, no claim of being exhaustive. We will mainly be concerned with the following settings:
In Part **1**, we consider Banach spaces X, Y, and form the Banach space $C(S, X)$ of all continuous functions $f : S \to X$, from the compact space S into X, equipped with the uniform norm, put $C(S, X) = C(S)$ if $X = \mathbb{R}$.
The main problem we will be concerned with is that of representing linear bounded operators $T : C(S, X) \to Y$ in an integral form:

$$f \in C(S, X),\ Tf = \int_S f\,d\mu$$

for some integration process with respect to a measure μ on the Borel $\sigma-$field $\mathcal{B}_S$ of S.
We give three major representation theorems**.** We start with the famous theorem of Bartle-Dunford Schwartz [1], representing general $X-$valued operators on $C(S)$. In this representation, the class of weakly compact operators has the important property of being represented by vector measures with values in the Banach space X.
Next we turn to Dinculeanu-Singer theorem [5], for general bounded operators $T : C(S, X) \to Y$. The theorem goes through the structure of the topological dual $C^*(S, X)$ of the function space $C(S, X)$, given in [24], and generalized in $[19]$. The concern of the third representation theorem is the construction of a class of operators from $C(S, X)$ into X, characterized by their Bochner form, given in $[16]$.
In Part **2**, the objective is to go beyond the Banach space setting, to a topological vector space (TVS) context. In this Part, X will be a locally convex space with dual X^* and S a locally compact space. We denote by $C_0(S, X)$ the function space of all continuous functions $f : S \to X$, vanishing outside a

compact set of S, put $C_0(S,X) = C_0(S)$ if $X = \mathbb{R}$. We are interested in representing linear bounded operators $T : C_0(S,X) \to X$, by means of weak integrals against scalar measure μ on S. First, in section **1,** we start with an operator $T : C_0(S,X) \to X$ and give conditions under which T can be written as a Pettis integral with respect to a scalar measure μ. Second, we consider the converse, that is, given a measure μ of bounded variation on $\mathcal{S}$, we seek for an operator $T : C_0(S,X) \to X$, which will have a Pettis integral form with respect to μ. This is a more delicate problem which needs additional assumptions on the space X. Solutions to this problem [18], under various conditions on the dual X^*, are given in section **2.**

Part 1

SOME REPRESENTATIONS THEOREMS IN BANACH SPACES

This Part is intended to present three main representations theorems for bounded operators in Banach spaces. For each of them, we give a short description of the integration process we use in the corresponding representation. These processes deal with functions and measures, each of them may be scalar or vector valued.

1. The Integral Representation of Bartle-Dunford-Schwartz

In what follows S will be an abstract set, $\mathcal{F}$ a $\sigma-$field of subsets of S.

1.1 Integration of a scalar function against a vector measure

1. Definition: Let X be a Banach space, $\mu : \mathcal{F} \to X$ a set function on $\mathcal{F}$ with values in X. We say that μ is a vector measure if for every pairwise disjoint sequence of sets $\{A_n\}$ in $\mathcal{F}$, the series $\sum\limits_n \mu(A_n)$ is unconditionally convergent in X and we have $\mu\left(\underset{n}{\cup} A_n\right) = \sum\limits_n \mu(A_n)$. A set function with this property is said to be $\sigma-$additive.

2 Definition: The semi-variation of the vector measure μ is defined by the set function:

$$(2) \qquad E \in \mathcal{F} \quad , \|\mu\|(E) = Sup \left| \sum_{i=1}^{n} \epsilon_i \ \mu(E_i) \right|$$

the supremum being taken over all finite partitions $\{E_i\}$ of E in $\mathcal{F}$, and all finite systems of scalars$\{\epsilon_i\}$ with $|\epsilon_i| \leq 1$.

The concepts of $\mu-$null sets and convergence $\mu-$almost everywhere are pertaining to the set function $\|\mu\|$. See $[7, \text{Chap.IV.10.8}]$, for details.
The semi-variation so defined is needed for some estimations in the integration process which will be used.

3. A simple measurable function of S into $\mathbb{R}$ is a function of the form
$f(\bullet) = \sum\limits_{i=1}^{n} \alpha_i \, \chi_{A_i}(\bullet)$, where χ_{A_i} is the characteristic function of the set A_j and where the A_j are pairwise disjoint sets in $\mathcal{F}$.

4. Definition: A function $f : S \to \mathbb{R}$ is measurable if and only if it is the limit $\mu-$almost everywhere of a sequence of simple measurable functions.

Now let X be a Banach space, and let $\mu : \mathcal{F} \to X$ be a vector measure. If $f : S \to \mathbb{R}$ is a real measurable function, we define the integral of f with respect to the vector measure μ and give some of its properties needed in integral representation. First we consider simple functions.

5. Definition: Let $f(\bullet) = \sum_{i=1}^{n} \alpha_i \, \chi_{A_i}(\bullet)$ be a simple measurable function. The integral of f with respect to μ over the set $E \in \mathcal{F}$ is defined by:

$$\int_E f \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(E \cap A_i)$$

just as in the customary real case, this integral does not depend on the representation of f.

It is clear that the integral so defined is linear as a function of f, and $\sigma-$additive as a set function of E. Moreover if $M = \sup_{s \in E} |f(s)|$ then:

$$\begin{aligned} \left\| \int_E f \, d\mu \right\| = \left\| M \sum_{i=1}^{n} \left(\frac{\alpha_i}{M} \right) \mu(E \cap A_i) \right\| &\leq M \sum_{i=1}^{n} \left| \frac{\alpha_i}{M} \right| \|\mu(E \cap A_i)\| \\ &\leq M \, \|\mu\|(E) \end{aligned}$$

so we deduce that:

$$\left\| \int_E f \, d\mu \right\| \leq \left(\sup_{s \in E} |f(s)| \right) \, \|\mu\|(E) \, .$$

6. Definition: A measurable function $f : S \to \mathbb{R}$ is said to be $\mu-$integrable, if there is a sequence f_n of simple functions such that:

(a) f_n converges to f $\mu-$almost everywhere

(b) The sequence $\left\{ \int_E f_n \, d\mu \right\}$ converges in the norm of X for each $E \in \mathcal{F}$.

The limit of the sequence $\left\{ \int_E f_n \, d\mu \right\}$ in (b) is called the integral of f with respect to μ over E and is denoted by $\int_E f \, d\mu$.

The integral so defined does not depend on the sequence f_n chosen. This fact is not trivial at all, see the proof in [7, IV.10.8]. On the other hand, it is straightforward that the integral $\int_E f \, d\mu$ is linear in f.

We record some properties of this integral in the following theorem:

7. Theorem: (a). *If f is bounded $\mu-$almost everywhere on the set E, then f is $\mu-$integrable over E and :*

$$\left\| \int_E f \, d\mu \right\| \leq \left(\sup_{s\in E} |f(s)| \right) . \|\mu\| (E) .$$

(b). *Let T be a linear bounded operator from X into the Banach space Y. Then $T\mu$ is a $Y-$valued vector measure on $\mathcal{F}$, and for any $\mu-$integrable f and any $E \in \mathcal{F}$ we have* $T\left(\int_E f \, d\mu \right) = \int_E f \, dT\mu.$

1.2 The Integral Representation of Bartle-Dunford-Schwartz

Let S be a compact space equipped with its Borel $\sigma-$field $\mathcal{B}_S$ and let us form the Banach space$C(S)$ of all continuous functions $f : S \to \mathbb{R}$, with the sup. norm.
The symbol X^* means the topological dual of the Banach space X and X^{**} its bidual. We will denote by $\mathcal{L}(X, E)$ the space of all linear bounded operators from X into the Banach space E. Recall the Banach space $r\sigma bv(\mathcal{B}_S)$ of all regular real measures with bounded variation on S.
We know from Riesz theorem that $r\sigma bv(\mathcal{B}_S)$ is isometrically isomorphic to the dual $C^*(S)$ of $C(S)$.
The following theorem is the general version of the representation of Bartle-Dunford-Schwartz:

8. Theorem: *Let $T : C(S) \to X$. be a bounded operator from $C(S)$ into the Banach space X. Then there exists a unique set function $\mu : \mathcal{B}_S \to X^{**}$ such that:*

(a) *$\mu(\bullet)\, x^* \in r\sigma bv(\mathcal{B}_S)$, for each $x^* \in X^*$.*
(b) *the mapping $x^* \to \mu(\bullet)\, x^*$ from X^* into $r\sigma bv(\mathcal{B}_S)$ is weak* continuous with respect to the topologies $\sigma(X^*, X)$ on X^* and $\sigma(C^*(S), C(S))$ on $C^*(S)$.*
(c) $x^*Tf = \int_S f(s)\, d\mu(s)\, x^*, f \in C(S), x^* \in X^*.$
(d) *$\|T\| = \|\mu\|(S)$, the semi-variation of μ at S (2).*

Conversely, if μ is a set function from $\mathcal{B}_S$ into X^{**} satisfying (a) and (b), then equation (c) defines an operator $T : C(S) \to X$ with norm given by (d), and such that $T^*x^* = \mu(\bullet)\, x^*$, $x^* \in X^*$.

Proof: In what follows we make use of the identication between the dual $C^*(S)$ of $C(S)$ and the Banach space $r\sigma bv(\mathcal{B}_S)$ of all regular real measures with bounded variation. Fix E in $\mathcal{B}_S$. and define $\varphi_E : C^*(S) \to \mathbb{R}$, by $\varphi_E(\lambda) = \lambda(E)$, $\lambda \in C^*(S)$. It is clear that $|\varphi_E(\lambda)| \leq |\lambda|$ (variation norm), so that $\varphi_E \in C^{**}(S)$ and $\|\varphi_E\| \leq 1$. Next, define $\mu : \mathcal{B}_S \to X^{**}$ by $\mu(E) = T^{**}(\varphi_E)$, $E \in \mathcal{B}_S$. It is easy to check that μ is additive.

Moreover if $T^* : X^* \to C^*(S)$ is the adjoint of T, then for $x^* \in X^*$, $T^*x^* = \lambda_{x^*} \in C^*(S)$ and $\lambda_{x^*}(E) = \varphi_E(\lambda_{x^*}) = \varphi_E(T^*(x^*)) = T^{**}\varphi_E\, x^*$. From the definition of μ, we deduce that $\lambda_{x^*}(E) = \mu(E)\, x^*$ and $T^*x^* = \mu(\bullet)\, x^*$. This proves (a) and (b), where (b) comes from the fact that the adjoint T^* is weak* continuous. By Riesz theorem $T^*x^*(f) = \int\limits_S f\, d\mu(\bullet)\, x^*$ and, since $T^*x^*(f) = x^*T(f)$, we get (c). Finally a straightforward computation gives (d).
Conversely, let μ be a set function from $\mathcal{B}_S$ into X^{**} satisfying (a) and (b). Then for each $f \in C(S)$, the mapping $x^* \to \int\limits_S f\, d\mu(\bullet)\, x^*$ of X^* into $\mathbb{R}$ is $\sigma(X^*, X)$-continuous; consequently there is a unique vector x_f in X such that $\int\limits_S f\, d\mu(\bullet)\, x^* = x^*(x_f)$. Define $T : C(S) \to X$ by $Tf = x_f$, $f \in C(S)$. It is not difficult to check that T satisfies (c) and (d).$\blacksquare$

Now it is natural to ask when does the set function μ take its values in $\gamma(X)$, where $\gamma : X \to X^{**}$ is the canonical embedding. As we will see presently, this will be true if the operator T is weakly compact.

9. Definition: Let X, Y be Banach spaces. A linear bounded operator $T : X \to Y$ is said to be weakly compact if for any bounded set B in X, the weak closure of TB is compact in the weak topology of Y.

Let us note the following facts about weakly compact operators([7, VI, 4.2,4.8])

10. Theorem: *A linear bounded operator $T : X \to Y$ is weakly compact if and only if $T^{**}X^{**}$ is contained in the natural embedding $\gamma(Y)$ of Y into Y^{**}.*

11. Theorem: *An operator T is weakly compact if and only if its adjoint T^* is weakly compact.*

Now we are in a position to state the integral representation of a weakly compact operator.

12. Theorem: *Let $T : C(S) \to X$. be a weakly compact operator from $C(S)$ into a Banach space X. Then there exists a unique vector measure $\lambda : \mathcal{B}_S \to X$ such that:*

(a') $x^*\lambda \in r\sigma bv(\mathcal{B}_S)$, for each $x^* \in X^*$.

(b') $Tf = \int\limits_S f\, d\lambda$, for all $f \in C(S)$

(c') $T^*x^* = x^*\lambda$, for each $x^* \in X^*$.

(d') $\|T\| = \|\lambda\|(S)$, *the semi-variation of λ at S.*

Conversely if λ is a vector measure on $\mathcal{B}_S$ with values in X, satisfying (a'), then the operator T defined by (b') is a weakly compact operator from $C(S)$

into X whose adjoint is given by $\left(c^{'}\right)$ *and whose norm is given by* $\left(d^{'}\right)$.

Proof: Let $\mu : \mathcal{B}_S \to X^{**}$ be the set function introduced in theorem **8** by $\mu(E) = T^{**}(\varphi_E)$, $E \in \mathcal{B}_S$. Since T is weakly compact, $T^{**}X^{**}$ is contained in the natural embedding $\gamma(X)$ of X into X^{**} by theorem **10.** Therefore $\mu(E) \in \gamma(X)$ for all $E \in \mathcal{B}_S$ and this allows to define $\lambda : \mathcal{B}_S \to X$ by $\lambda(E) = \gamma^{-1}\mu(E)$. Then $\lambda(E) \in X$ and for $x^* \in X^*$: $x^*\lambda(E) = x^*\gamma^{-1}\mu(E) = \gamma\left(\gamma^{-1}\mu(E)\right)x^* = \mu(E)\,x^*$, where the second equality comes from the definition of γ. This shows that $x^*\lambda \in r\sigma bv(\mathcal{B}_S)$, for each $x^* \in X^*$. By Pettis theorem [7 chap.IV], λ is a vector measure on $\mathcal{B}_S$ into X. From theorem **8** (c) we get

$$x^*Tf = \int_S f(s)\,d\mu(s)\;x^* = \int_S f(s)\,dx^*\lambda(s),\; f \in C(S), x^* \in X^*.$$

Now apply theorem **7** (b) to get $x^*Tf = x^*\int_S f(s)\,d\lambda(s)$, for each $x^* \in X^*$. This yields $Tf = \int_S f\,d\lambda$, by the Hahn-Banach theorem, so $\left(b^{'}\right)$ is proved. On the other hand we have $x^*Tf = \int_S f(s)\,dx^*\lambda(s) = T^*x^*(f)$, hence $T^*x^* = x^*\lambda$, by Riesz theorem, whence $\left(c^{'}\right)$. Finally $\left(d^{'}\right)$ is immediate from point (d) of theorem **8.**

Conversely if λ is a vector measure on $\mathcal{B}_S$ with values in X, satisfying $\left(a^{'}\right)$ and if T is defined by $\left(b^{'}\right)$, then T is linear bounded (theorem **7.**(a)) and for each $x^* \in X^*$ we have $T^*x^* = x^*\lambda$. By lemma 2.3 in [1], T^* is weakly compact and so, theorem **11**, T is weakly compact.■

Note that for a locally compact space S, theorem **12** has been extended by Kluvanek in [14, Lemma 2].

2. The Representation Theorem of Dinculeanu-Singer

2.1 Integration of a vector valued function against an operator valued measure

Let X, Y be Banach spaces, S a compact space and let $C(S, X)$ be the Banach space of all continuous functions $f : S \to X$, with the sup. norm:

$$f \in C(S,X),\; \|f\| = \underset{s\in S}{Sup}\,\|f(s)\|$$

Put $C(S,X) = C(S)$ if $X = \mathbb{R}$.

The symbols X^*, X^{**}have the meaning of section **1.2**. We will denote by $\mathcal{L}(X,E)$ the space of all linear bounded operators from X into the Banach space E. All the set functions considered here are, except otherwise stated, assumed to be defined on the Borel $\sigma-$field $\mathcal{B}_S$ of S.

1 We will deal with additive operator valued set functions

$$G : \mathcal{B}_S \rightarrow \mathcal{L}(X, E)$$

and for such functions, we define the semivariation by:

$$(a) \qquad B \in \mathcal{B}_S,\ \widetilde{G}(B) = Sup \left\| \sum_i G(A_i) .x_i \right\|$$

the supremum being taken over all finite partitions $\{A_i\}$ of B in $\mathcal{B}_S$ and all finite systems of vectors $\{x_i\}$ in X, with $\|x_i\| \leq 1$, $\forall i$.

2 To each additive set function $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, Y^{**})$ there is associated a family $\{G_{y^*},\ y^* \in Y^*\}$ of additive X^*-valued set functions given by:

$$A \in \mathcal{B}_S,\ G_{y^*}(A) .x = G(A) .x\ (y^*)$$

Note that since $G(A) \in \mathcal{L}(X, Y^{**})$, we have:
$\|G_{y^*}(A) .x\| \leq \|G(A) .x\| . \|y^*\| \leq \|G(A)\| . \|x\| . \|y^*\|$.
Moreover we have the important estimation:(see [5], Proposition 5 p.55)

$$(a)' \qquad A \in \mathcal{B}_S,\ \widetilde{G}(A) = Sup \{|G_{y^*}|(A),\ \ \|y^*\| \leq 1\}.$$

where $|G_{y^*}|$ is the variation of G_{y^*} as given in the following definition

3 Definition: If $\lambda : \mathcal{B}_S \rightarrow E$ is a vector measure, the variation of λ is defined by the positive set function $|\lambda|(\bullet)$ given by:

$$B \in \mathcal{B}_S,\ |\lambda|(B) = Sup \sum_i \|\lambda(A_i)\|$$

the supremum being taken over all finite partitions $\{A_i\}$ of B in $\mathcal{B}_S$.
We say that λ is of bounded variation if $|\lambda|(B) < \infty$, for all $B \in \mathcal{B}_S$ and in this case, $|\lambda|$ itself is a finite positive measure. λ is said to be regular if $|\lambda|$ is regular in the customary sense. Let us denote by $r\sigma bv(\mathcal{B}_S, E)$ the vector space of all regular $E-$valued vector measures with bounded variation, put $r\sigma bv(\mathcal{B}_S, E) = r\sigma bv(\mathcal{B}_S)$, if $S = \mathbb{R}$. Then we have:

4 Theorem: *The variation $|\lambda|$ defines a norm on the space $r\sigma bv(\mathcal{B}_S, E)$, and for this norm, $r\sigma bv(\mathcal{B}_S, E)$ is a Banach space. Moreover, in the specific case $E = X^*$, we have:*

$$B \in \mathcal{B}_S,\ |\lambda|(B) = Sup \left| \sum_i \lambda(A_i) .x_i \right|$$

the supremum being taken over all finite partitions $\{A_i\}$ *of* B *in* $\mathcal{B}_S$ *and all finite systems of vectors* $\{x_i\}$ *in* X, *with* $\|x_i\| \leq 1$, $\forall i$.
In other words, for X^*-valued vector measures, the variation is equal to the semivariation (see **2.** (a)) .

Now the aim is to define the integral of a function f on S with values in X, against an additive $\mathcal{L}(X, E)-$valued set function G on $\mathcal{B}_S$. The result of this integration will be a vector of the Banach space E. First, we define the measurable functions which will be integrated.

5 Definition: Let $f : S \to X$ be a function on S with values in the Banach space X, f is measurable if it is the uniform limit of a sequence of simple measurable functions.

We denote by $M(S, X)$ the set of all measurable functions $f : S \to X$. It is clear that $M(S, X)$ is a vector space. Moreover, in the Banach space of all bounded functions from S into X, with the uniform norm, $M(S, X)$ is the closure of the subspace of all simple measurable functions.

6 Definition: Let G be an additive $\mathcal{L}(X, E)-$valued set function on $\mathcal{B}_S$. If $f(\bullet) = \sum_{i=1}^{n} x_i.\chi_{A_i}(\bullet)$ is a simple measurable function, the integral of f on the set $B \in \mathcal{B}_S$ with respect to G is defined by the formula:

$$\int_B f \, dG = \sum_{i=1}^{n} G(A_i \cap B) . x_i$$

From the additivity of G, it is easy to see that the integral does not depend on the form of f, and is linear. Moreover if $M = \sup_{s \in B} f \|(s)\|$, then:

$\left\| \int_B f \, dG \right\| = M \left\| \sum_{i=1}^{n} G(A_i \cap B) . \frac{x_i}{M} \right\| \leq M.\widetilde{G}(B)$. So we get for each simple f and each $B \in \mathcal{B}_S$:

$$(*) \qquad \left\| \int_B f \, dG \right\| \leq \left(\sup_{s \in B} \|f(s)\| \right) .\widetilde{G}(B)$$

where $\widetilde{G}(B)$ is the semivariation defined in $\mathbf{2}(a)$**.**

7 Definition: Let $f : S \to X$ be a measurable function, i.e $f \in M(S, X)$. We define the integral of f over B with respect to G by the limit,

$\int_B f \, dG = \lim_n \int_B f_n \, dG$, where f_n is any sequence of simple measurable functions converging uniformly to f. The limit is in the norm of X.
To check that the integral is well defined, let f_n, g_n be two sequences of simple measurable functions converging uniformly to f. Then by the inequality

6 $(*)$, we have:

$$\left\| \int_B f_n \, dG - \int_B g_n \, dG \right\| = \left\| \int_B (f_n - g_n) \, dG \right\| \leq \left(\sup_{s \in B} \| f_n(s) - g_n(s) \| \right) \widetilde{G}(B)$$
$$\longrightarrow 0, \quad n \longrightarrow \infty.$$

8 Proposition: *For each measurable function f and each $B \in \mathcal{B}_S$, we have:*

$$\left\| \int_B f \, dG \right\| \leq \left(\sup_{s \in B} \| f(s) \| \right) . \widetilde{G}(B)$$

Proof: Let f_n be a sequence of simple measurable functions converging uniformly to f. If $\epsilon > 0$, there is an integer $N \geq 1$ such that for all $n \geq N$, we have $\left\| \int_B f \, dG - \int_B f_n \, dG \right\| < \epsilon$. So we deduce that for all $n \geq N$:

$$\left\| \int_B f \, dG \right\| \leq \left\| \int_B f \, dG - \int_B f_n \, dG \right\| + \left\| \int_B f_n dG \right\| < \epsilon + \left(\sup_{s \in B} \| f_n(s) \| \right) . \widetilde{G}(B).$$

Letting $n \longrightarrow \infty$ and then $\epsilon \searrow 0$, we get the inequality.■

In some representation theorems, we need to integrate continuous functions $f : S \to X$. This will be possible according to the following proposition. Recall that $C(S, X)$ is the Banach space of all continuous functions $f : S \to X$, with the sup. norm.

9 Proposition: *The Banach space $C(S, X)$ is a closed subspace of $M(S, X)$.*

Proof: See Proposition **1, § 19** in [5].■

2.2 On the Dual Space $\mathbf{C}_0^*(S, X)$

In the next representation theorem, we need the structure of the topological dual $C^*(S, X)$ of the Banach space $C(S, X)$. This is given by the following theorem:

10 Theorem:[19] *Let S be a locally compact space and let $C_0(S, X)$ be the Banach space of all continuous functions $f : S \to X$ vanishing at infinity. Then there is an isometric isomorphism between the Banach spaces $C_0^*(S, X)$ and $r\sigma bv(\mathcal{B}_S, X^*)$, where to the functional U in $C_0^*(S, X)$ corresponds the measure λ in $r\sigma bv(\mathcal{B}_S, X^*)$ via the formula:*

$$f \in C_0(S, X), \quad U(f) = \int_S f \, d\lambda$$
$$\|U\| = |\lambda|$$

*where the integral is the one defined in section **2.1**.*

Now we turn to the one of the most general representation theorem.

2.3 The Representation Theorem of Dinculeanu-Singer

11 Theorem: *Every linear bounded operator* $T : C(S, X) \to Y$ *determines a unique set function* $G : \mathcal{B}_S \to \mathcal{L}(X, Y^{**})$ *such that:*

(i) *G is finetely additive and with finite semivariation* $\widetilde{G}$.
(ii) *The set function* G_{y^*} *is a vector measure in* $r\sigma bv(\mathcal{B}_S, X^*)$ *for each* $y^* \in Y^*$.
(iii) *The function* $y^* \to G_{y^*}$ *is weak** *continuous with the* $\sigma(Y^*, Y)$*-topology on* Y^* *and the* $\sigma(C^*(S,X), C(S,X))-$*topology on* $C^*(S,X)$.
(iv) $Tf = \int_S f\, dG,\ f \in C(S,X)$, *which really means that* $\gamma Tf = \int_S f\, dG$, *where* $\gamma : Y \to Y^{**}$ *is the canonical embedding.*
(v) $\|T\| = \widetilde{G}(S)$ (semivariation of G defined in $\mathbf{1}(a)$.).
(vi) $T^* y^* = G_{y^*}$, *for all* $y^* \in Y^*$.

Proof: Consider the adjoint operator $T^* : Y^* \to C^*(S,X)$. From theorem **2.1,** for each $y^* \in Y^*$, $T^* y^*$ is a vector measure on $\mathcal{B}_S$ with values in X^*, which we will denote by μ_{y^*}. Thus we have for $f \in C(S,X)$ $T^* y^*(f) = y^* T f = \int_S f\, d\mu_{y^*}$, and $\|T^* y^*\| = |\mu_{y^*}|$
Next fix $x \in X$, $A \in \mathcal{B}_S$ and define $G(A)x : Y^* \to \mathbb{R}$ by $G(A)x(y^*) = \mu_{y^*}(A)(x)$. Then it is easy to check that $G(A)x \in Y^{**}$ and that we have $\|G(A)x\| \le \|T\| . \|x\|$. This allows us to define, for A fixed in $\mathcal{B}_S$, $G(A) : X \to Y^{**}$ by $x \to G(A)x$. Then $G(A)$ is linear and bounded with $\|G(A)\| \le \|T\|$, for all $A \in \mathcal{B}_S$. Furthermore, the set function $A \to G(A)$ from $\mathcal{B}_S$ into $\mathcal{L}(X, Y^{**})$ is additive and satisfies:
$G(A)x(y^*) = G_{y^*}(A)(x) = \mu_{y^*}(A)(x)$, $A \in \mathcal{B}_S$, $y^* \in Y^*$, $x \in X$.
Moreover we have, from $\mathbf{2.}(a)$, $\widetilde{G}(S) = Sup\{\upsilon(G_{y^*}, S),\ \|y^*\| \le 1\}$. So we deduce that $\widetilde{G}(S) = Sup\{\upsilon(\mu_{y^*}, S),\ \|y^*\| \le 1\}$.
Since $\|T^* y^*\| = |\mu_{y^*}| = \upsilon(\mu_{y^*}, S)$, we get
$$\widetilde{G}(S) = Sup\{\|T^* y^*\|,\ \|y^*\| \le 1\} = \|T^*\| = \|T\|.$$
Let us observe that:

- (i) is satisfied by the definition of G.
- (ii) is a consequence of the relation $G_{y^*} = \mu_{y^*}$.
- (vi) is true since $T^* y^* = \mu_{y^*} = G_{y^*}$.
- (iii) is valid by (vi) and the weak*continuity of the adjoint.
- (v) is proved by the computation above involving $\widetilde{G}(S)$.

It remains to prove (iv). To this end, we use the integration process described in section **2.1** above. Consider the space $M(S,X)$ of measurable functions $f : S \to X$. By Proposition **9,** we have $C(S,X) \subset M(S,X)$, so $\int_S f\, dG$ is well defined for $f \in C(S,X)$, and is in Y^{**}, since G takes its values in $\mathcal{L}(X,Y^{**})$. Put for a moment $Uf = \int_S f\,dG$ and observe that for $y^* \in Y^*$, $Uf(y^*) = \int_S f\,dG_{y^*}$ (check the formula for f simple and extend to all $f \in M(S,X)$, using standard methods.). But we have also $T^*y^*(f) = y^*Tf = \int_S f\,dG_{y^*}$, hence $y^*Tf = Uf(y^*)$, for all $f \in C(S,X)$ and $y^* \in Y^*$. Since $Tf \in Y$, we have $y^*Tf = \gamma Tf(y^*)$, where $\gamma : Y \to Y^{**}$ is the canonical embedding. consequently $\gamma Tf(y^*) = Uf(y^*)$, and then $\gamma Tf = Uf$ for all $f \in C(S,X)$, proving (iv).■

12 Remark: The set function G above is not $\sigma-$additive in general. To get $\sigma-$additivity, needs additional assumption on the operator T. It has been proved by Dorakov in [6], that the representing measure G of the operator $T : C(S,X) \to Y$ has all its values in $\mathcal{L}(X,Y)$ if and only if for each $x \in X$ the operator $T_x : C(S) \to Y$ defined by $T_x f = T(x.f)$, $f \in C(S)$, is weakly compact. In this case, G is $\sigma-$additive in the strong operator topology of $\mathcal{L}(X,Y)$.

3. Representation of Bounded Operators by Bochner Integral

3.1 Integration of a vector valued function against a scalar measure: The Bochner Integral

Let $(S,\mathcal{F},\mu)$ be a measure space, with μ a finite positive measure. We will assume that $(S,\mathcal{F},\mu)$ is complete. As before, X will be a Banach space with topological dual X^*. In this section we need to integrate functions $f : S \to X$, with respect to the scalar measure μ.
First we need measurability. For all details on Bochner integral, see[10].

1 Definition: (i).An elementary measurable function $f : S \to X$ is a function of the form $f(\bullet) = \sum_i x_i.\chi_{A_i}(\bullet)$, where $\{A_i\}$ is a countable partition of S in $\mathcal{F}$ and $\{x_i\}$ a sequence of vectors in X. We denote by $\mathcal{E}$ the set of all elementary measurable functions $f : S \to X$.

(ii).A function $f : S \to X$ is said to be strongly measurable if there is a sequence of elementary measurable functions f_n converging $\mu-$almost everywhere to f. Let $\mathcal{M}$ be the set of all strongly measurable functions $f : S \to X$.

(iii).A function $f : S \to X$ is said to be weakly measurable if for each $x^* \in X^*$, the real function $x^* \circ f : S \to \mathbb{R}$ is measurable.

The relation between the two types of measurability, weak-strong, this is given by the following theorem of Pettis:

2 Theorem: *A function $f : S \to X$ is strongly measurable if and only if the following conditions are satisfied:*

(a). f is weakly measurable

(b). There is a set $S_0 \in \mathcal{F}$ such that $\mu(S \setminus S_0) = 0$ and the image $f(S_0)$ of S_0 by f is separable.

In particular, if X is a separable Banach space, the weak and strong measurability are equivalent.

3 Definition: We say that the elementary measurable function $f(\bullet) = \sum\limits_i x_i.\chi_{A_i}(\bullet)$ is $\mu-$integrable if $\sum\limits_i \|x_i\| .\mu(A_i) < \infty$. In this case we define the integral of f with respect to μ by $\int\limits_S f\,d\mu = \sum\limits_i x_i.\mu(A_i)$. Likewise the integral of f over the set $E \in \mathcal{F}$ is $\int\limits_E f\,d\mu = \sum\limits_i x_i.\mu(A_i \cap E)$.

The following theorem gives one of the outstanding facts about the Bochner integral.

4 Theorem: *A function $f : S \to X$ is $\mu-$integrable if and only if f is strongly measurable and $\int\limits_S \|f\|\,d\,\mu < \infty$.*

We denote by $\mathcal{L}_1(\mu, X)$ the set of all $\mu-$integrable functions.

5 Now we extend the definition of the Bochner integral to a real measure μ, with bounded variation, by $\int\limits_E f\,d\mu = \int\limits_E f\,d\mu^+ - \int\limits_E f\,d\mu^-$, where μ^+, μ^- are the positive and negative parts of μ.

For any real μ, it is customary to denote by $L_1(\mu, X)$ the vector space of all equivalence classes, with respect to the equality $|\mu| - a.e$, of $\mu-$integrable functions.If $f \in L_1(\mu, X)$ we put $\|f\|_1 = \int\limits_S \|f\|\,d\,\mu$. Then we have:

6 Theorem: *$\|\bullet\|_1$ is a norm which makes $L_1(\mu, X)$ a Banach space. Moreover we have $\left\|\int\limits_E f\,d\mu\right\| \leq \int\limits_E \|f\|\,d\mu$, for all $E \in F$ and $f \in L_1(\mu, X)$.*

The operator $I_\mu : L_1(\mu, X) \to X$ given by $I_\mu(f) = \int\limits_S f\,d\mu$ is linear continuous with norm $\|I_\mu\| = |\mu|$.

Proof: Mimic the classical proof for the real space $L_1(\mu)$.$\blacksquare$

7 Theorem: *If Y is a Banach space and if $T : X \to Y$ is a linear bounded operator, then for each function $f \in L_1(\mu, X)$ the function Tf is in $L_1(\mu, Y)$ and we have* $T\left(\int\limits_E f\, d\mu\right) = \int\limits_E Tf\, d\mu$.

3.2 Bounded Operators and Bochner Integral

In this section, we introduce a class of bounded operators $T : C(S, X) \to X$, which admit a representation by Bochner integral with respect to a scalar measure on $\mathcal{B}_S$. For the construction of the Bochner integral and its properties, we refer the reader to [10].

8 Let μ be a scalar measure with bounded variation on $\mathcal{B}_S$ and let us consider the operator $I_\mu : C(S, X) \to X$ introduced jn theorem **8**

$$f \in C(S, X), \, I_\mu f = \int\limits_S f \, d\, \mu$$

where the integral is in the sense of Bochner.
For each $x^* \in X^*$ let $U_{x^*} : C(S, X) \to C(S)$ be the linear bounded operator given by $U_{x^*} f = x^* \circ f$, where $(x^* \circ f)(s) = x^*(f(s))$, $f \in C(S, X)$, $s \in S$.
We collect some facts about U_{x^*} for later use:

9 Proposition: *(i) For each $x^* \in X^*$, we have $\|U_{x^*}\| = \|x^*\|$. Moreover there exists $z^* \in X^*$ such that for every $h \in C(S)$, there is a solution $f \in C(S, X)$ of the equation $U_{z^*} f = h$ with $\|f\| = \|h\|$.*
(ii)If $V : C(S) \to \mathbb{R}$ is linear and bounded then we have:
$\|V\| = Sup\{\|V \circ U_{x^*}\|, \ \|x^*\| \le 1\}$.

Proof: The proposition is a consolidated form of lemmas 2.2, 2.3 in [16].■

All what we need about I_μ is the following:

10 Proposition: *(i) I_μ is a linear bounded operator from $C(S, X)$ into X. Moreover, if $U : X \to E$, is a bounded operator from X into the Banach space E, then we have $U(I_\mu f) = I_\mu(Uf)$, for all $f \in C(S, X)$, where Uf is the function in $C(S, E)$ given by $(Uf)(s) = U(f(s))$, for $s \in S$.*
(ii) $\|I_\mu\| = |\mu|$ (the variation of μ).

Proof: comes from theorems **8** and **9**..■

Now let us consider a bounded operator $T : C(S, X) \to X$, and ask the question of the existence of a scalar measure μ on $\mathcal{B}_S$ such that $Tf = I_\mu f$, for all $f \in C(S, X)$. In what follows we introduce a class of operators $T : C(S, X) \to X$ for which this problem does have a positive answer.

11 Definition:[16] Let C_{XX} be the class of linear bounded operators
$T : C(S, X) \to X$ which satisfy the following condition:

$$(C) \quad x^*, y^* \in X^*, \;\; f, g \in C(S,X): \; x^* \circ f = y^* \circ g \Longrightarrow x^*Tf = y^*Tg$$

In some sense, the operators of the class C_{XX} preserve the continuous functionals of X. It is easy to check that C_{XX} is a closed subspace of the Banach space $\mathcal{L}(C(S,X), X)$, endowed with the uniform norm. Also let us observe that for each scalar measure μ with bounded variation on $\mathcal{B}_S$, the operator I_μ is in C_{XX} (take $E = \mathbb{R}$ and $U = U_{x^*}$ in Proposition **10** (i)).

The outstanding fact about the class C_{XX} is contained in the following theorem which will be essential for the integral representation.

12 Theorem: *There is an isometric isomorphism between the Banach space C_{XX} and the topological dual $C^*(S)$ of $C(S)$, for each non trivial Banach space X. In other words, there exists a linear bijective mapping $\varphi : C_{XX} \to C^*(S)$ such that $\|\varphi(T)\| = \|T\|$, for all $T \in C_{XX}$.*

Proof: We show how to construct the mapping $\varphi : C_{XX} \to C^*(S)$, with the given properties. Let $T \in C_{XX}$ and $h \in C(S)$. By Proposition **9**(i) there is $z^* \in X^*$ and $f \in C(S,X)$ such that $U_{z^*}f = h$ and $\|f\| = \|h\|$. Next define $V : C(S) \to \mathbb{R}$, $V(h) = z^*Tf$. Then V is well defined because of condition (C) imposed to the operator T; moreover V is linear and bounded, i.e $V \in C^*(S)$. Let us define $\varphi : C_{XX} \to C^*(S)$, by $\varphi(T) = V$. It is clear from this construction that φ is linear. Furthermore we have:

$$(*) \qquad \forall x^* \in X^*, \; V \circ U_{x^*} = x^* \circ T$$

Indeed, for $f \in C(S,X)$ and $x^* \in X^*$, $U_{x^*}f$ is in $C(S)$, so by **9**(i) there exists $g \in C(S,X)$ such that $U_{x^*}f = x^* \circ f = z^* \circ g$. Therefore:

$$\begin{aligned} V \circ U_{x^*}(f) = V(x^* \circ f) &= z^* \circ Tg \quad \text{(by the definition of } V) \\ &= x^* \circ Tf \quad \text{(by condition } C) \end{aligned}$$

since f is arbitrary, $(*)$ is proved.
On the other hand, we have from **9**(ii), $\|V\| = Sup\{\|V \circ U_{x^*}\|, \; \|x^*\| \leq 1\}$, and using $(*)$ we get $\|V\| = Sup\{\|x^* \circ T\|, \; \|x^*\| \leq 1\} = \|T^*\| = \|T\|$.
This proves that $\|V\| = \|\varphi(T)\| = \|T\|$, that is φ is an isometry.
To achieve the proof, we construct $\theta : C^*(S) \to C_{XX}$, which will be the inverse of φ. If $V \in C^*(S)$, then, by Riesz theorem, there exists a bounded real measure μ on $\mathcal{B}_S$ such that $Vh = \int_S h \, d\mu$, and $\|V\| = |\mu|$, for all $h \in C(S)$. Then, define θ by $\theta(V) = I_\mu$. We know that $I_\mu \in C_{XX}$, so θ is well defined and is an isometry from $C^*(S)$ into C_{XX}, since $\|\theta(V)\| = \|I_\mu\| = |\mu| = \|V\|$. We prove that θ is the inverse of φ. Let $T \in C_{XX}$, with $\varphi(T) = V \in C^*(S)$. Let μ be the measure associated to V as before. By definition we have $\theta(V)f = I_\mu(f) = \int_S f \, d\mu$ and for each $x^* \in X^*$:

$$x^*\theta(V)f = x^* \int_S f \, d\mu = \int_S x^* \circ f \, d\mu$$

$$= V\left(x^* \circ f\right)$$
$$= x^* T f \qquad \text{(from } (*)\ V = \varphi(T)\text{)}$$

since x^* is arbitrary, we deduce from Hahn-Banach theorem that $\theta(V) f = Tf$ and consequently $\theta \circ \varphi(T) = T$.
Similarly we have $\varphi \circ \theta(V) = V$, all $V \in C^*(S)$.■

As a consequence of this theorem, we give a representation of an operator in the class C_{XX} by mean of a Bochner integral.

13 Theorem: *Let $T{:}C(S,X) \rightarrow X$ be an operator in the class C_{XX}. Then there is a unique bounded real measure μ on $\mathcal{B}_S$ such that :*

(i) $T(f) = \int\limits_S f\, d\mu$, *for all* $f \in C(S,X)$.

(ii) $\|T\| = |\mu|$.

Proof: We use the transformations φ and θ introduced in the proof of theorem **12.** Put $\varphi(T) = V \in C^*(S)$, and $Vh = \int\limits_S h\, d\mu$, for $h \in C(S)$. Appealing to the relation $(*)$ used in the proof of theorem **12**, we get: $V(x^* \circ f) = x^* \circ Tf$

$$= \int\limits_S x^* \circ f\, d\mu = x^* \int\limits_S f\, d\mu, \quad f \in C(S,X) \text{ and } x^* \in X^*.$$

consequently $x^* \circ Tf = x^* \int\limits_S f\, d\mu$, for all $x^* \in X^*$,

which gives $Tf = \int\limits_S f\, d\mu$.■

Part 2

INTEGRAL REPRESENTATIONS IN TOPOLOGICAL VECTOR SPACES

Now, the objective is to go beyond the Banach space setting, to a TVS context. The aim of this Chapter is to get integral representation theorems of bounded operators by weak integrals, in an appropriate framework of TVS.

In all this chapter, unless otherwise stated, S will be a topological space and μ a real measure of bounded variation on the Borel $\sigma-$field $\mathcal{B}_S$. Also X will be a locally convex Hausdorff space with topological dual X^*, and for $\theta \in X^*, x \in X$, we denote by $\langle \theta, x \rangle$ the functional duality between X and X^*.

1. Integral Representation by Pettis Integral

Suppose that S is a locally compact space and let X be a locally convex TVS. We denote by $C_0(S,X)$ the set of all continuous functions $f : S \to X$, vanishing outside a compact set of S, put $C_0(S,X) = C_0(S)$ if $X = \mathbb{R}$. We are interested in representing linear bounded operators $T : C_0(S,X) \to X$, by means of weak integrals against scalar measures on the Borel $\sigma-$field $\mathcal{B}_S$ of S. Before handling more closely this problem, we need some preliminary facts about the space $C_0(S,X)$.

1.1 Topological preliminaries: If K is a compact set in S, let $C(S,K,X)$ be the set of all continuous functions $f : S \to X$, vanishing outside K. It is clear that $C(S,K,X)$ is a linear subspace of $C_0(S,X)$. We equip $C(S,K,X)$ with the topology τ_K generated by the family of seminorms:

$$(a) \qquad f \in C(S,K,X), \quad \widetilde{p}_{\alpha,K} = \underset{t\in K}{Sup}\, p_\alpha(f(t))$$

where $\{p_\alpha\}$ is the family of seminorms generating the locally convex topology of X. The topology τ_K is the topology of uniform convergence on K.

Next let us observe that $C_0(S,X) = \bigcup_K C(S,K,X)$, the union being performed over all the compact subsets K of S. On the other hand if $K_1 \subset K_2$, then the natural embedding $i_{K_1K_2} : C(S,K_1,X) \to C(S,K_2,X)$ is continuous. This allows one to provide the space $C_0(S,X)$ with the inductive topology τ induced by the subspaces $C(S,K,X), \tau_K$. The facts we need about the space $C_0(S,X), \tau$ are well known:

1 Proposition: (a) *The space $C_0(S,X), \tau$ is locally convex Hausdorff and for each compact K, the relative topology of τ on $C(S,K,X)$ is τ_K, i.e the canonical embedding $i_K : C(S,K,X) \to C_0(S,X)$ is continuous.*

(b) *Let $T : C_0(S,X) \to V$ be a linear operator of $C_0(S,X)$ into the locally convex Hausdorff space V, then T is continuous if and only if the restriction $T \circ i_K$ of T to the subspace $C(S,K,X)$ is continuous for each compact K.*

2 Definition: For each θ in the topological dual X^* of X and for each function $f \in C_0(S,X)$, define the function $U_\theta f$ on S by $U_\theta f(s) = \theta(f(s)) = \langle \theta, f(s) \rangle$. Then U_θ sends $C_0(S,X)$ into $C_0(S)$. Recall that $C_0(S)$ is equipped with the uniform norm.

3 Lemma: *The operator U_θ is linear and bounded. Moreover for each $\theta \neq 0$, U_θ is onto.*

Proof: First it is clear that $U_\theta f \in C_0(S)$. Now by Proposition **1**(b), we have to show that for each compact set $K \subset S$ the operator $U_\theta \circ i_K : C(S,K,X) \to C_0(S)$ is bounded. Since θ is bounded, there is a seminorm p_α on X and a constant M such that $|\theta(x)| \leq M\ p_\alpha(x)$ for all $x \in X$. So we have $|\theta(f(s))| \leq M\ p_\alpha(f(s))$ if $f \in C(S,K,X)$, and $U_\theta \circ i_K(f)(s) = \theta(f(s))$, $s \in S$; it follows that

$\|U_\theta f\| = \underset{s \in K}{Sup} |\theta(f(s))| \leq M.\underset{s \in K}{Sup}\, p_\alpha(f(s))$. Since by formula (a), the right side of this inequality is $M\ \widetilde{p}_{\alpha,K}(f)$, we deduce that U_θ is continuous. Now suppose $\theta \neq 0$. Then there exists $x \in X$ such that $x \neq 0$ and $\theta(x) \neq 0$. It is clear that we can assume $\theta(x) = 1$. Now let $h \in C_0(S)$ and define $f : S \to X$ by $f(t) = h(t).x$, then $f \in C_0(S,X)$ and we have $U_\theta(f)(s) = U_\theta(h(s)x) = h(s)$, because $\theta(x) = 1$. It follows that U_θ is onto.■

In what follows we deal with the representation of bounded operators $T : C_0(S,X) \to X$, by weak integrals in the sense of the definition:

4 Definition: We say that a bounded operator $T : C_0(S,X) \to X$, has a Pettis integral form if there exist a scalar measure of bounded variation μ on $\mathcal{B}_S$ such that, for every continuous functional θ in X^*, we have:

$$f \in C_0(S,X),\ \ \langle \theta, Tf \rangle = \int_S \langle \theta, f \rangle\, d\mu$$

5 Definition: Let us denote by $\mathcal{P}$ the class of all bounded operators $T : C_0(S,X) \to X$ satisfying the following condition:

$$(I)\ \ \text{For } \theta, \sigma \in X^* \text{ and } f, g \in C_0(S,X),\ \ \text{if } U_\theta f = U_\sigma g \quad \text{then } \theta(Tf) = \sigma(Tg)$$

Condition (I) in this context, is analogous to condition (C) in Definition **11,** section **3**, Part **1.**

It is easy to check that $\mathcal{P}$ is a subspace of the space $\mathcal{L}(C_0(S,X), X)$ of all bounded operators from $C_0(S,X)$ to X. Also one can prove that $\mathcal{P}$ is closed in the weak operator topology of $\mathcal{L}(C_0(S,X), X)$. Note also that for a given bounded $T : C_0(S,X) \to X$, Definition **4** implies condition (I) i.e $T \in \mathcal{P}$. The crucial point is that condition (I) implies the Pettis integral form of Definition **4**, for some bounded scalar measure μ on $\mathcal{B}_S$. To prove this fact, the following theorem is basic:

6 Theorem: *Let $T : C_0(S,X) \to X$ be a bounded operator satsfying condition (I). Then there exists a unique bounded functional φ on $C_0(S)$ such that:*

$$(b) \qquad \forall \theta \in X^*, \quad \varphi \circ U_\theta = \theta \circ T.$$

Proof: Let h be fixed in $C_0(S)$. If $0 \neq \theta \in X^*$, by lemma **3** there is $f \in C_0(S,X)$, solution of $U_\theta f = h$. Define $\varphi(h) = \theta(Tf)$. Then φ is well defined since if $U_\theta f = U_\sigma g = h$, for $\theta, \sigma \in X^*$ and $f, g \in C_0(S,X)$, then $\theta(Tf) = \sigma(Tg)$. It is clear that φ is linear. Also (b) is immediate by construction. It remains to prove that φ is bounded. Let $h \in C_0(S)$ and let $0 \neq \theta \in X^*$;

since every solution f of $U_\theta f = h$ works in the definition of φ, we may choose, $f(t) = h(t)x$, with x fixed in X so that $\theta(x) = 1$. In this case we have,see .**1.1** :

$$(c) \qquad \widetilde{p}_{\gamma,K}(f) = \|h\| . p_\gamma(x)$$

where K is the support of f (= support of h), and p_γ is a seminorm on X. By formula (b) $\varphi(h) = \theta(Tf)$,and, since θ is bounded, there is a constant $M > 0$ and a seminorm p_α on X such that: $|\varphi(h)| = |\theta(Tf)| \leq M.p_\alpha(Tf)$. But T is bounded; so for each compact $K \subset S$ and for the preceding p_α, there is a constant $\lambda > 0$ and a seminorm $\widetilde{p}_{\beta,K}$ on $C(S,K,X)$ such that:
$p_\alpha(Tf) \leq \lambda . \widetilde{p}_{\beta,K}(f)$. Appealing to formula (c), with $\gamma = \beta$, we get:
$p_\alpha(Tf) \leq \lambda . \|h\| . p_\beta(x)$. Now, with the above estimation of $|\varphi(h)|$, we deduce that $|\varphi(h)| \leq M.\lambda.p_\beta(x).\|h\|$, which proves the boundedness of φ. Uniqueness comes from (b) since U_θ is onto.■

As a consequence we have the main representation theorem:

1.2 Integral Representation by Pettis Integral:

7 Theorem: *Let $T : C_0(S,X) \to X$ be in the class $\mathcal{P}$. Then there is a unique bounded signed measure μ on $\mathcal{B}_S$ such that $\langle \theta, Tf \rangle = \int_S \langle \theta, f \rangle \, d\mu$ holds for all θ in X^* and $f \in C_0(S,X)$. Moreover for each seminorm p_α on X we have $|T|_{p_\alpha} = |\mu|$, where $|\mu|$ is the total variation of μ and $|T|_{p_\alpha}$ is the $p_\alpha-$norm of T defined by $|T|_{p_\alpha} = Sup\left\{ p_\alpha(Tf) : f \in \widetilde{B}_{p_\alpha} \right\}$ with $\widetilde{B}_{p_\alpha} = \left\{ f \in C_0(S,X) : \underset{S}{Sup}\, p_\alpha(f(s)) \leq 1 \right\}$.*

Proof: Let φ be given by theorem **6**. By the Riesz representation theorem [22], there exists a unique bounded signed measure μ on $\mathcal{B}_S$ such that:

$$(d) \quad \forall h \in C_0(S) \quad \varphi(h) = \int_S h(s) \, d\mu(s)$$

Taking h of the form $h = U_\theta f = \langle \theta, f(\bullet) \rangle$, with $f \in C_0(S,X)$ and citing formula (b) again, yields $\varphi \circ U_\theta f = \theta \circ Tf = \int_S \langle \theta, f(s) \rangle \, d\mu(s)$, which means

$\langle \theta, Tf \rangle = \int_S \langle \theta, f \rangle \, d\mu$. Now to compute $|T|_{p_\alpha}$, observe from the integral form of $\theta \circ Tf$ that $|\theta \circ Tf| \leq Sup\{|\langle \theta, f(s) \rangle| : s \in S\} . |\mu|$. Taking the supremum in both sides over $\theta \in B^o_{p_\alpha}$, the polar set of the unit ball $B_{p_\alpha} = \{x \in X, \quad p_\alpha(x) \leq 1\}$ of X, we get:

$$\begin{aligned}
\underset{\theta \in B^o_{p_\alpha}}{Sup} |\theta \circ Tf| = p_\alpha(Tf) &\leq \underset{\theta \in B^o_{p_\alpha}}{Sup} \ \underset{s \in S}{Sup} |\langle \theta, f(s) \rangle| . |\mu| \\
&= \underset{s \in S}{Sup} \ \underset{\theta \in B^o_{p_\alpha}}{Sup} \ |\langle \theta, f(s) \rangle| . |\mu| = Sup \ p_\alpha(f(s)) . |\mu| \\
&\leq |\mu|, \text{ for } f \in \widetilde{B}_{p_\alpha}.
\end{aligned}$$

So we deduce that $|T|_{p_\alpha} \leq |\mu|$. To see the reverse inequality, let us consider a function $f \in C_0(S, X)$ of the form $f = g.x$, with $g \in C_0(S)$ satisfying $\|g\| \leq 1$ and x fixed in X such that $p_\alpha(x) = 1$. With this choice, the function f belongs to the unit ball $\widetilde{B}_{p_\alpha}$. Then we have $\langle \theta, f(s) \rangle = g(s) . \theta(x)$ and

$$\begin{aligned}
\langle \theta, Tf \rangle &= \textstyle\int_S \ \langle \theta, f(s) \rangle \ d\mu(s) \\
&= \theta(x) \textstyle\int_S \ g(s) \ d\mu(s);
\end{aligned}$$

so that $\underset{\theta \in B^o_{p_\alpha}}{Sup} |\theta \circ Tf| = p_\alpha(Tf) = p_\alpha(x) \left| \int_S \ g(s) \ d\mu(s) \right| = \left| \int_S \ g(s) \ d\mu(s) \right|$,

since $p_\alpha(x) = 1$. So we get $p_\alpha(Tf) = \left| \int_S \ g(s) \ d\mu(s) \right| \leq |T|_{p_\alpha}$ because $f \in \widetilde{B}_{p_\alpha}$.

Therefore $Sup \ \left\{ \left| \int_S \ g(s) \ d\mu(s) \right|, \ g \in C_0(S), \ \ \|g\| \leq 1 \right\} = |\mu| \leq |T|_{p_\alpha}$ ■.

By this theorem we may denote each operator T in the class $\mathcal{P}$ by the conventional symbol

$$(\mathbf{W}) \qquad f \in C_0(S, X), \quad Tf = P - \textstyle\int_S \ f(s) \ d\mu(s)$$

where the letter P stands for Pettis integral.

8 Remark: Usually a weak integral is defined as a vector x^{**} in the second conjugate space X^{**} (see the Dunford integral in [4]). The construction of the Pettis integral, that is a Dunford integral with values in X, is not so straightforward and needs additional conditions on the space X. In our present setting we were able to construct directly a whole class of Pettis integrals with values in the topological vector space X. In Section **2** below,we shall consider the reverse direction, that is, we start with a bounded measure μ on S and we will construct directly the Pettis integrals under μ by means of a bounded operator T in the class $\mathcal{P}$. But this will be achieved under additional assumptions on X.

2. Operators associated to scalar measures via Pettis integrals

In this section we start with a bounded scalar measure μ on $\mathcal{B}_S$ and we seek for a linear bounded $T : C_0(S,X) \to X$ such that the correspondence between μ and T would be given by formula $(\mathbf{W})$. First let us make some observations.

2.1 Operators via Pettis integrals

A little inspection of $(\mathbf{W})$ suggests the following quite plausible observations: First the integral $\int\limits_S \langle \theta, f(s) \rangle \, d\mu(s)$, as a linear functional of θ on X^*, should be at least continuous for some convenient topology on X^*. Also the existence of the corresponding Tf in $(\mathbf{W})$ will require that such topology on X should be compatible for the dual pair $\langle X^*, X \rangle$. Finally, to get the continuity of the functional $\theta \to \int\limits_S \langle \theta, f(s) \rangle \, d\mu(s)$, one can seek conditions such that if $\theta \to 0$ in an appropriate manner, then $\langle \theta, f(s) \rangle$ goes to 0 uniformly for $s \in S$. Since μ is bounded this will give $\int\limits_S \langle \theta, f(s) \rangle \, d\mu(s) \to 0$.

In what follows we shall show that such a program can be realized for a locally convex space having the convex compactness property (see Definition **11** below).

9 We shall denote by X^*_τ the dual space X^* equipped with the Mackey topology $\tau(X^*, X)$. According to Mackey–Arens theorem, it is the topology of uniform convergence on the family of absolutely convex $\sigma(X, X^*)-$compact sets of X. Also it is the largest compatible topology for the dual pair $\langle X^*, X \rangle$. Then we have the well known:

10 Proposition: *For each $x^{**} \in (X^*_\tau)^*$ there exists a unique $x \in X$ such that $x^{**}(\theta) = \theta(x)$, $\forall \theta \in X^*$.*

11 Definition: A locally convex space X is said to have the convex compactness property if for every compact set $K \subset X$, the absolute convex closure K_0 of K is also compact. For example, every quasicomplete locally convex space has the convex compactness property.

12 Theorem: *Let X be a locally convex space with the convex compactness property, and whose dual X^* is equipped with the Mackey topology $\tau(X^*, X)$. If μ is a bounded scalar measure on $\mathcal{B}_S$, then there is a unique bounded operator $T : C_0(S,X) \to X$ in the class $\mathcal{P}$ satisfying $(\mathbf{W})$, with $|T|_{p_\alpha} = |\mu|$ for each seminorm p_α on X.*

Proof: Fix f in $C_0(S,X)$ and define the functional $\Gamma_f : X^* \to \mathbb{R}$, by $\Gamma_f(\theta) = \int_S \langle \theta, f(s) \rangle \, d\mu(s)$. It is clear that Γ_f is linear. Moreover $\Gamma_f \in (X^*_\tau)^*$. Indeed it is enough to prove that $\lim\limits_{\theta \to 0} \Gamma_f(\theta) = 0$. If $\theta \to 0$, in X^*_τ, then for each absolutely convex $\sigma(X, X^*)-$compact $B \subset X$, $\theta(x) \to 0$ uniformly on B. But since $f \in C_0(S,X)$, the set $K = \{f(s) : s \in S\}$ is compact.

Therefore, by the convex compactness property we deduce that K_0, the absolutely convex closure of K is compact, hence weakly compact; so $\theta \to 0$ uniformly on K_0. Consequently $\langle \theta, f(s) \rangle \to 0$ uniformly in $s \in S$.
Hence $\Gamma_f(\theta) = \int_S \langle \theta, f(s) \rangle \, d\mu(s) \to 0$, because μ is bounded. We deduce that $\Gamma_f \in (X_\tau^*)^*$; by Proposition **2** there is a unique $\xi_f \in X$ such that $\Gamma_f(\theta) = \langle \theta, \xi_f \rangle, \forall \theta \in X^*$. Now let us define the operator $T : C_0(S,X) \to X$, by $Tf = \xi_f,\ f \in C_0(S,X)$. It is easily checked that T is linear, and satisfies $(\mathbf{W})$ by constuction. We have to show that T is bounded. Let p_α be a seminorm on X, and let K be a compact subset of X. For $f \in C(S,K,X)$, we have:$p_\alpha(\xi_f) =$

$$
\begin{aligned}
p_\alpha(Tf) &= \underset{\theta \in B^o_{p_\alpha}}{Sup} |\theta \circ Tf| \\
&= \underset{\theta \in B^o_{p_\alpha}}{Sup} \left| \int_S \langle \theta, f(s) \rangle \, d\mu(s) \right| \\
&\leq \underset{\theta \in B^o_{p_\alpha}}{Sup} \ \underset{s \in K}{Sup} |\langle \theta, f(s) \rangle| \, . \, |\mu| \\
&= \underset{s \in K}{Sup} \ \underset{\theta \in B^o_{p_\alpha}}{Sup} |\langle \theta, f(s) \rangle| \, . \, |\mu| \\
&= \widetilde{p}_{\alpha,K}(f) \, . \, |\mu|
\end{aligned}
$$

which proves the continuity of T. The relation $|T|_{p_\alpha} = |\mu|$ is proved as in **7**$\blacksquare$.

Theorems **7,** and **12** may be put together to give:

13 Theorem: *If X is a locally convex space having the convex compactness property, then there is an isometric isomorphism between the space $\mathcal{P}$ and the topological dual $C_0^*(S)$ of the space $C_0(S)$. In this isomorphism the operator $T \in \mathcal{P}$ corresponds to the measure $\mu \in C_0^*(S)$ via the integral representation:*

$$\forall \theta \in X^*, \ \forall f \in C_0(S,X), \ \langle \theta, Tf \rangle = \int_S \langle \theta, f \rangle \, d\mu, \qquad |T|_{p_\alpha} = |\mu| \, .$$

14 Remark: One essential point in the proof of Theorem **12** was the uniform convergence in $s \in S$ of $\langle \theta, f(s) \rangle$ to 0 when $\theta \to 0$ inX_τ^*. This has given $\Gamma_f \in (X_\tau^*)^*$. If we want to get rid of the convex compactness condition, we must have $\Gamma_f(\theta) \to 0$ when $\theta \to 0$ inX_τ^*. But if $\theta \to 0$ inX_τ^*, we certainly have $\langle \theta, f(s) \rangle \to 0$ for each $s \in S$. Then, as the set $\{\langle \theta, f(s) \rangle, \ s \in S\}$ is bounded for each $\theta \in X^*$, getting uniform convergence with respect to s is reminiscent to a uniform boundedness principle, which in the present setting, should be valid for the dual X_τ^* of X. It is well known that a general version of this principle has been stated, via equicontinuity, for the barrelled topological vector spaces, [27, Chapter 9, Theorem 9.3.4]. With this observation in mind we can state:

15 Theorem: *Let X be a locally convex Hausdorff space whose dual X_τ^* is a barrelled space. If μ is a bounded signed measure on $\mathcal{B}_S$, then there is a unique bounded operator $T : C_0(S,X) \to X$ in the class $\mathcal{P}$ satisfying $(\mathbf{W})$ with respect to μ and such that $|T|_{p_\alpha} = |\mu|$.*

Proof: Consider the set of the linear functionals on X_τ^*,$\mathcal{F} = \{\langle \bullet, f(s) \rangle, \ s \in S\}$. Since for each θ the function $s \to \langle \theta, f(s) \rangle$ is continuous and since $f \in C_0(S,X)$, we deduce that for each $\theta \in X_\tau^*$ the set $\{\langle \theta, f(s) \rangle, \ s \in S\}$ is bounded, that, is the family $\mathcal{F}$ is pointwise bounded. Since X_τ^* is barrelled, the family $\mathcal{F}$ is equicontinuous, by the uniform boundedness principle.Therefore if $\theta \to 0$ inX_τ^*, for each $\epsilon > 0$ there is a 0-neighborhood V in X_τ^* such that if $\theta \in V$ we have $|\langle \theta, f(s) \rangle| \le \epsilon$, for all $s \in S$. This means that $\langle \theta, f(s) \rangle \to 0$ uniformly in $s \in S$. This gives the continuity of the functional $\Gamma_f(\theta) = \int_S \langle \theta, f(s) \rangle \, d\mu(s)$. Now the proof goes along the same lines as in the proof of Theorem **12**.$\blacksquare$

Remark: Problems similar to those considered in this paper can be taken in a more general setting. One can go beyond locally convex TVS, to general TVS (e.g non locally convex spaces), but things are not so easy to deal with, due to the structure of such spaces and their duals. In this case difficulties arise from the fact that new integration processes on different function spaces are involved for an appropriate attempt to the problem. Attempts of this kind has been done by different authors on some special function spaces in [8], [12], [23], [26]

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